Lower Bounds of Concurrence for Tripartite Quantum Systems

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Abstract

We derive an analytical lower bound for the concurrence of tripartite quantum mixed states. A functional relation is established relating concurrence and the generalized partial transpositions.

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As the key physical resources in quantum information processing and quantum computation [1], the quantum entangled states have been investigated with a great deal of effort in the past years [2-14]. So far for generic mixed states only partial solutions are known on detection and quantification of entanglement in an operational way. Concurrence is one of the well defined quantitative measures of entanglement. For two-qubit case another measure, entanglement of formation [15, 16] is a monotonically increasing function of concurrence and an elegant formula of concurrence was derived analytically by Wootters in [4], which plays an essential role in describing quantum phase transition in various interacting quantum many-body systems [17] and may affect macroscopic properties of solids significantly [18]. What is more, it can be experimentally measured [19].

Nevertheless, calculation of the concurrence is a formidable task for higher dimensional case. Therefore some nice algorithms and progresses have been concentrated on possible lower bounds of the concurrence for qubit-qudit systems [11, 12] and for bipartite systems in arbitrary dimensions [5, 14] but involving numerical optimization over a large number of free parameters. In [20] an analytical lower bound of concurrence for any dimensional mixed bipartite quantum states has been presented, which is further shown to be exact for some special classes of states and detects many bound entangled states.

Although the lower bound for entanglement of formation can be similarly investigated for bipartite case [21], for tripartite case the entanglement of formation is not yet well defined. In contract, the concurrence for tripartite states is well defined. In this paper we consider the lower bound of concurrence for tripartite states, by exploring the connection between the generalized partial transposition (GPT) criterion and concurrence.

Let $\mathcal{H}_A$, $\mathcal{H}_B$ and $\mathcal{H}_C$ be three finite dimensional Hilbert spaces associated with the subsystems A, B and C, with dimensions $dim\ A = m$, $dim\ B = n$ and $dim\ C = p$. The concurrence for a general pure tripartite state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ is defined by

$$C(|\psi\rangle) = \sqrt{3 - \mathrm{Tr}(\rho_A^2 + \rho_B^2 + \rho_C^2)}, \tag{1}$$

where the reduced density matrix ρ_A (resp. ρ_B, ρ_C) is obtained by tracing over the subsystems B and C (resp. A and C, A and B). The concurrence for a tripartite mixed state ρ is defined by the convex roof,

$$C(\rho) \equiv \min_{\{p_i,|\psi_i\rangle\}} \sum_i p_i C(|\psi_i\rangle), \tag{2}$$

for all possible ensemble realizations $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, where $|\psi_i\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, $p_i \geq 0$ and $\sum_i p_i = 1$. For any pure product state $|\psi\rangle$, $C(|\psi\rangle)$ vanishes according to the definition. Consequently, if a state ρ is *separable*, then $C(\rho) = 0$.

To get a lower bound of (2), we relate directly the concurrence to the generalized partial transposition separability criterion. We first recall some notations used in various matrix operations [9, 22, 23].

A generic matrix M can be always written as $M = \sum_{i,j} a_{ij} \langle j| \otimes |i\rangle$, where $|i\rangle, |j\rangle$ are vectors of a suitably selected normalized real orthogonal basis. We define the operations $\mathcal{T}_r$ (resp. $\mathcal{T}_c$) to be the row transposition (resp. column transposition) of M which transposes the second (resp. first) vector in the above tensor product expression of M:

$$\mathcal{T}_r(M) = \sum_{i,j} a_{ij} \langle j| \otimes \langle i|, \qquad \mathcal{T}_c(M) = \sum_{i,j} a_{ij} |j\rangle \otimes |i\rangle. \tag{3}$$

It is easily verified that $\mathcal{T}_c\mathcal{T}_r(M) = \mathcal{T}_r\mathcal{T}_c(M) = M^t$, where t denotes matrix transposition.

We further define $\mathcal{T}_{r_k}$ (resp. $\mathcal{T}_{c_k}$) ($k = A, B, C, AB, BC, AC$) to be the row (resp. column) transpositions with respect to the subsystems k. Set $\mathcal{T}_{\{x_1,x_2,...\}} \equiv \mathcal{T}_{x_1}\mathcal{T}_{x_2}...$ for $x_1,\ x_2 \subset \Gamma \equiv \{r_A, c_A, r_B, c_B, r_C, c_C, r_{AB}, r_{AC}, r_{BC}, c_{AB}, c_{AC}, c_{BC}\}$. We consider the generalized partial transposition operations on a tripartite density matrix given by $\mathcal{T}_\mathcal{Y}$, where $\mathcal{T}_\mathcal{Y}$ stands for all partial transpositions contained in $\mathcal{Y}$ which is a subset of Γ. The GPT criterion says that if a tripartite $m \times n \times p$ density matrix is separable, then the trace norm $||\rho^{\mathcal{T}_\mathcal{Y}}|| \leq 1$, where $\rho^{\mathcal{T}_\mathcal{Y}} = \mathcal{T}_\mathcal{Y}(\rho)$, for instance $\rho^{\mathcal{T}_{\{c_A, r_B, r_C\}}} \equiv \mathcal{T}_{\{c_A\}}\mathcal{T}_{\{r_B\}}\mathcal{T}_{\{r_C\}}(\rho)$ and so on. In the following we discuss three classes of $\mathcal{Y}$:

I: $\mathcal{Y}_i = \{c_k, r_k\}$, where $i = 1, 2, 3$ for $k = A, B, C$ respectively;

II: $\mathcal{Y}_4 = \{c_A, r_{BC}\}$, $\mathcal{Y}_5 = \{c_{AB}, r_C\}$, $\mathcal{Y}_6 = \{c_{AC}, r_B\}$;

III: $\mathcal{Y}_7 = \{c_A, r_B\}$, $\mathcal{Y}_8 = \{c_A, r_C\}$, $\mathcal{Y}_9 = \{c_B, r_C\}$.

It is verified that $\rho^{\mathcal{T}_{\mathcal{Y}_i}} = \rho^{T_k}$, where $k = A, B, C$ with respect to $i = 1, 2, 3$, T_k stands for partial transposition with respect to the subsystem k. Hence the operations in class I correspond to the partial transpositions of ρ. While the operations in class III correspond to the realignments of a tripartite state ρ [9, 10].

We first study the relation between GPT and the concurrence for three qubits ($m = n = p = 2$). A three-qubit state $|\Psi\rangle$ can be written in terms of the generalized Schmidt

decomposition [24],

$$|\Psi\rangle \ = \ \lambda_0|000\rangle + \lambda_1 e^{i\psi}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle \tag{4}$$

with normalization condition $\lambda_i \geq 0, \ \ 0 \leq \psi \leq \pi$, where $\sum_i \mu_i = 1$, $\mu_i \equiv \lambda_i^2$.

Defining $\Delta \equiv |\lambda_1\lambda_4 e^{i\psi} - \lambda_2\lambda_3|^2$, we have, for $\rho = |\Psi\rangle\langle\Psi|$

$$Tr\rho_A^2 = 1 - 2\mu_0(1 - \mu_0 - \mu_1),$$

$$Tr\rho_B^2 = 1 - 2\mu_0(1 - \mu_0 - \mu_1 - \mu_2) - 2\Delta,$$

$$Tr\rho_C^2 = 1 - 2\mu_0(1 - \mu_0 - \mu_1 - \mu_3) - 2\Delta.$$

Therefore

$$C^2(\rho) = 2\mu_0(3 - 3\mu_0 - 3\mu_1 - \mu_2 - \mu_3) + 4\Delta, \tag{5}$$

which varies smoothly from 0, for pure product states, to $\frac{3}{2}$ for maximally entangled pure states.

On the other hand, we have

$$\rho^{\mathcal{T}y_1} = \left(\begin{array}{cccccccc}
\mu_0 & 0 & 0 & 0 & \lambda_0\lambda_1 e^{i\psi} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_0\lambda_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_0\lambda_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \lambda_0\lambda_4 & 0 & 0 & 0 \\
\lambda_0\lambda_1 e^{-i\psi} & \lambda_0\lambda_2 & \lambda_0\lambda_3 & \lambda_0\lambda_4 & \mu_1 & \lambda_2\lambda_1 e^{i\psi} & \lambda_3\lambda_1 e^{i\psi} & \lambda_4\lambda_1 e^{i\psi} \\
0 & 0 & 0 & 0 & \lambda_1\lambda_2 e^{-i\psi} & \mu_2 & \lambda_2\lambda_3 & \lambda_2\lambda_4 \\
0 & 0 & 0 & 0 & \lambda_3\lambda_1 e^{-i\psi} & \lambda_3\lambda_2 & \mu_3 & \lambda_3\lambda_4 \\
0 & 0 & 0 & 0 & \lambda_4\lambda_1 e^{-i\psi} & \lambda_4\lambda_2 & \lambda_4\lambda_3 & \mu_4
\end{array}\right).$$

As $\rho^{\mathcal{T}y_1} = {\rho^{\mathcal{T}y_1}}^\dagger$, the square root of the eigenvalues of $\rho^{\mathcal{T}y_1}{\rho^{\mathcal{T}y_1}}^\dagger$ is the absolute value of the eigenvalues of $\rho^{\mathcal{T}y_1}$: $\{0, 0, 0, 0, \pm\sqrt{\mu_0(\mu_2 + \mu_3 + \mu_4)}, \frac{1}{2}(1 \pm \sqrt{1 - 4\mu_0(\mu_2 + \mu_3 + \mu_4)})\}$. Therefore the norm of $\rho^{\mathcal{T}y_1}$ is given by

$$||\rho^{\mathcal{T}y_1}|| = 1 + 2\sqrt{\mu_0(\mu_2 + \mu_3 + \mu_4)}. \tag{6}$$

Similarly we have

$$||\rho^{\mathcal{T}y_2}|| \ = \ 1 + 2\sqrt{\Delta + \mu_0(\mu_3 + \mu_4)}, \tag{7}$$

$$||\rho^{\mathcal{T}y_3}|| \ = \ 1 + 2\sqrt{\Delta + \mu_0(\mu_2 + \mu_4)}. \tag{8}$$

A lower bound for the concurrence of three-qubit states is given by the following theorem.

[Theorem 1]. For any three-qubit mixed quantum state ρ, the concurrence $C(\rho)$ satisfies

$$C(\rho) \geq max\left\{\|\rho^{\mathcal{T}_{y_i}}\| - 1,\ \frac{1}{\sqrt{2}}(\|\rho^{\mathcal{T}_{y_j}}\| - 1)\right\}, \tag{9}$$

where $i = 1, 2, 3,\ j = 4, 5, 6$.

[Proof]. Let us assume that one has already found an optimal decomposition $\sum_i p_i\rho^i$ for ρ to achieve the infimum of $C(\rho)$, where ρ^i are pure state density matrices. Then $C(\rho) = \sum_i p_i C(\rho^i)$ by definition. Noticing that $\|\rho^{\mathcal{T}_{y_j}}\| \leq \sum_i p_i \|(\rho^i)^{\mathcal{T}_{y_j}}\|$, for all possible j, due to the convex property of the trace norm, one only needs to show $C(\rho^i) \geq (\|(\rho^i)^{\mathcal{T}_{y_j}}\| - 1)$ for $j = 1, 2, 3$ and $C(\rho^i) \geq \frac{1}{\sqrt{2}}(\|(\rho^i)^{\mathcal{T}_{y_j}}\| - 1)$, for $j = 4, 5, 6$.

For a pure state ρ^i, from Eqs. (5) , (6), (7) and (8), we have

$$C^2(\rho^i) - (\|(\rho^i)^{\mathcal{T}_{y_1}}\| - 1)^2 = 2\mu_0\mu_4 + 4\Delta \geq 0,$$

$$C^2(\rho^i) - (\|(\rho^i)^{\mathcal{T}_{y_2}}\| - 1)^2 = 4\mu_0\mu_2 + 2\mu_0\mu_4 \geq 0$$

and

$$C^2(\rho^i) - (\|(\rho^i)^{\mathcal{T}_{y_3}}\| - 1)^2 = 4\mu_0\mu_3 + 2\mu_0\mu_4 \geq 0.$$

That is $C(\rho^i) \geq (\|(\rho^i)^{\mathcal{T}_{y_j}}\| - 1)$ for $j = 1, 2, 3$.

For a pure state ρ^i, we consider it as a $2 \otimes 4$, or $4 \otimes 2$ bipartite state, respectively. From the results for bipartite systems [20], we have

$$\begin{aligned} 1 - Tr((\rho^i_A)^2) &\geq \frac{1}{2}(||(\rho^i)^{\mathcal{T}_{\{c_A, r_{BC}\}}}|| - 1)^2, \\ 1 - Tr((\rho^i_B)^2) &\geq \frac{1}{2}(||(\rho^i)^{\mathcal{T}_{\{c_{AC}, r_B\}}}|| - 1)^2, \\ 1 - Tr((\rho^i_C)^2) &\geq \frac{1}{2}(||(\rho^i)^{\mathcal{T}_{\{c_{AB}, r_C\}}}|| - 1)^2. \end{aligned}$$

Therefore

$$\begin{aligned} &C(\rho^i) = \sqrt{3 - Tr((\rho^i_A)^2) - Tr((\rho^i_B)^2) - Tr((\rho^i_C)^2)} \geq \\ &\frac{1}{\sqrt{2}} max\left\{(||(\rho^i)^{\mathcal{T}_{\{c_A, r_{BC}\}}}|| - 1), (||(\rho^i)^{\mathcal{T}_{\{c_{AC}, r_B\}}}|| - 1), (||(\rho^i)^{\mathcal{T}_{\{c_{AB}, r_C\}}}|| - 1)\right\}, \end{aligned} \tag{10}$$

i.e.

$$C(\rho^i) \geq \frac{1}{\sqrt{2}} max\left\{||(\rho^i)^{\mathcal{T}_{y_j}}|| - 1\right\}, \quad j = 4, 5, 6,$$

which ends the proof. □

As an example, let us consider the Dür-Cirac -Tarrach states [25]:

$$\rho = \sum_{\sigma=\pm} \lambda_0^\sigma |\Psi_0^\sigma\rangle\langle\Psi_0^\sigma| + \sum_{j=1}^{3} \lambda_j (|\Psi_j^+\rangle\langle\Psi_j^+| + |\Psi_j^-\rangle\langle\Psi_j^-|), \tag{11}$$

where the orthonormal Greenberger-Horne-Zeilinger (GHZ)-basis

$$|\Psi_j^\pm\rangle \equiv \frac{1}{\sqrt{2}}(|j\rangle_{AB}|0\rangle_C \pm |(3-j)\rangle_{AB}|1\rangle_C),$$

$|j\rangle_{AB} \equiv |j_1\rangle_A |j_2\rangle_B$ with $j = j_1 j_2$ in binary notation. For example, $|\Psi_0^\pm\rangle \equiv \frac{1}{\sqrt{2}}(|000\rangle \pm |111\rangle)$ is the standard GHZ states.

A direct calculation gives rise to $||\rho^{\mathcal{T}_{\mathcal{Y}_1}}|| = \frac{4}{3}$, $||\rho^{\mathcal{T}_{\mathcal{Y}_2}}|| = ||\rho^{\mathcal{T}_{\mathcal{Y}_3}}|| = 1$, and $||\rho^{\mathcal{T}_{\mathcal{Y}_j}}|| = 0.8727$ for $j = 4, 5, 6$. Therefore, $C(\rho) \geq \frac{1}{3}$ according to theorem 1 for $\lambda_0^+ = \frac{1}{3}; \lambda_1 = \lambda_3 = \frac{1}{6}; \lambda_0^- = \lambda_2 = 0$. This shows that the state is entangled, which is also a conclusion implied by [25, 26].

We have obtained lower bounds of the concurrence in terms of the generalized partial transposition. Similar to the bipartite case, it is also possible to find lower bounds of the concurrence in terms of the realignment operations, described in class III, which correspond to the realignments of the density matrix ρ on A, B; A, C and B, C subsystems, while leaving the remaining C; B and A subsystems unchanged. For instance, with respect to the operation $\mathcal{Y}_7$, $\rho^{\mathcal{T}_{\{c_A, r_B\}}}$ implies $\rho_{ijm,kln}^{\mathcal{T}_{\{c_A, r_B\}}} = \rho_{ikm,jln}$, where the indices $i(k, m)$ and $j(l, n)$ are viewed as the row and column indices for the subsystem $A(B, C)$ respectively.

Let us consider a special-type of three-qubit states by setting $\lambda_i = 0$, $i = 1, 2, 3$ in (4),

$$|\Phi\rangle \ = \ \lambda_0|000\rangle + \lambda_4|111\rangle \tag{12}$$

with normalization condition $\lambda_0,\ \lambda_4 \geq 0$, $\lambda_0^2 + \lambda_4^2 = 1$. We get, for $\rho_0 = |\Phi\rangle\langle\Phi|$,

$$\rho_0^{\mathcal{T}_{\{c_A, r_B\}}} = \lambda_0^2|000\rangle\langle000| + \lambda_0\lambda_4|010\rangle\langle011| + \lambda_0\lambda_4|101\rangle\langle100| + \lambda_4^2|111\rangle\langle111|.$$

Hence the sum of its singular values is $||\rho_0^{\mathcal{T}_{\{c_A, r_B\}}}|| \ = \ 1 + 2\lambda_0\lambda_4$. Similarly, we have $||\rho_0^{\mathcal{T}_{\{c_A, r_C\}}}|| = ||\rho_0^{\mathcal{T}_{\{c_B, r_C\}}}|| = 1 + 2\lambda_0\lambda_4$.

From Eqs. (5), (6), (7) , (8) and direct calculations we have at last $C(\rho_0) = \sqrt{6\mu_0\mu_4}$, $||\rho_0^{\mathcal{T}_{\mathcal{Y}_j}}|| = 2\lambda_0\lambda_4 + 1$, $j = 1, ..., 9$. By using the procedure in proving Theorem 1, we arrive at:

[Corollary]. For any three-qubit mixed state with decomposition $\rho = \sum_i p_i|\Psi_i\rangle\langle\Psi_i|$, if $|\Psi_i\rangle$ can be written in the form (12) for any i, then the concurrence $C(\rho)$ satisfies

$$C(\rho) \geq max\{||\rho^{\mathcal{T}_{\mathcal{Y}_j}}||\} - 1, \quad j = 1, ..., 9. \tag{13}$$

We remark that once a density matrix has a decomposition with all the pure states of the form (12), then its all other possible decompositions will also have the form (12), since other decompositions can be obtained from the unitarily linear combinations of this decomposition, and any linear combinations of the type (12) still have the form (12).

Although for general three-qubit states we do not have an analytical relation between concurrence and the realignment operations (Class III), the numerical computations imply that $C(\rho) \geq max\{||\rho^{\mathcal{T}_{\mathcal{Y}_j}}||\} - 1$, $j = 7, 8, 9$, is still valid. We chose 10^6 random vectors $(\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \psi)$ for state (4), calculated $C(\rho)$ and $||\rho^{\mathcal{T}_{\mathcal{Y}_j}}|| - 1$, $j = 7, 8, 9$. All results agree with the inequality in Corollary. From the proof of Theorem 1, it implies that the inequality would be also correct for mixed states.

Generalizing the results of Theorem 1 to arbitrary dimensional tripartite quantum states, we have the following lower bounds:

[**Theorem 2**]. For any $m \otimes n \otimes p$ $(m \leq n, p)$ tripartite mixed quantum state ρ, the concurrence $C(\rho)$ satisfies

$$C(\rho) \geq max\left\{\sqrt{\frac{1}{m(m-1)}}(||\rho^{\mathcal{T}_{\mathcal{Y}_a}}|| - 1), \sqrt{\frac{1}{q(q-1)}}(||\rho^{\mathcal{T}_{\mathcal{Y}_b}}|| - 1), \sqrt{\frac{1}{r(r-1)}}(||\rho^{\mathcal{T}_{\mathcal{Y}_c}}|| - 1)\right\}, \tag{14}$$

where $q = min(n, mp)$ and $r = min(p, mn)$, $\mathcal{Y}_a = \mathcal{Y}_1$ or $\mathcal{Y}_4$, $\mathcal{Y}_b = \mathcal{Y}_2$ or $\mathcal{Y}_6$, $\mathcal{Y}_c = \mathcal{Y}_3$ or $\mathcal{Y}_5$.

[**Proof**]. Let us assume that one has already found an optimal decomposition $\sum_i p_i \rho^i$ for ρ to achieve the infimum of $C(\rho)$, where ρ^i are pure state density matrices. Then $C(\rho) = \sum_i p_i C(\rho^i)$ by definition. Noticing that $||\rho^{\mathcal{T}_{\mathcal{Y}_k}}|| \leq \sum_i p_i ||(\rho^i)^{\mathcal{T}_{\mathcal{Y}_k}}||$, $k = a, b, c$, due to the convex property of the trace norm, one only needs to show $C(\rho^i) \geq \sqrt{\frac{1}{j(j-1)}}(||(\rho^i)^{\mathcal{T}_{\mathcal{Y}_k}}|| - 1)$, where $j = m$, q, r for $k = a$, b, c, respectively. For a pure state ρ^i, we consider it as a $m \otimes np$, $n \otimes mp$ or $mn \otimes p$ bipartite state, respectively. From the result of [20], we obtain

$$1 - Tr((\rho^i_A)^2) \geq \frac{1}{m(m-1)}(||(\rho^i)^{\mathcal{T}_{\mathcal{Y}_a}}|| - 1)^2,$$
$$1 - Tr((\rho^i_B)^2) \geq \frac{1}{q(q-1)}(||(\rho^i)^{\mathcal{T}_{\mathcal{Y}_b}}|| - 1)^2,$$
$$1 - Tr((\rho^i_C)^2) \geq \frac{1}{r(r-1)}(||(\rho^i)^{\mathcal{T}_{\mathcal{Y}_c}}|| - 1)^2.$$

Therefore from the definition of $C(\rho^i)$,

$$\sqrt{3 - Tr((\rho^i_A)^2) - Tr((\rho^i_B)^2) - Tr((\rho^i_C)^2)} \geq$$
$$max\left\{\sqrt{\frac{1}{m(m-1)}}(||\rho^{\mathcal{T}_{\mathcal{Y}_a}}|| - 1), \sqrt{\frac{1}{q(q-1)}}(||\rho^{\mathcal{T}_{\mathcal{Y}_b}}|| - 1), \sqrt{\frac{1}{r(r-1)}}(||\rho^{\mathcal{T}_{\mathcal{Y}_c}}|| - 1)\right\}. \tag{15}$$

□

In summary, by making a novel connection with the generalized partial transpositions, we have provided an entirely analytical formula for lower bound of concurrence for tripartite systems. One only needs to calculate the trace norm of certain matrices, which avoids complicated optimization procedure over a large number of free parameters in numerical approaches. The results could be used to indicate possible quantum phase transitions in condensed matter systems, and to analyze finite size or scaling behavior of entanglement in various interacting quantum many-body systems. In principle one can similarly investigate the lower bound for general multipartite quantum systems. However as the generalized Schmidt decomposition of multipartite pure states becomes more complicated when the number of subsystems increases, the problem would be more sophisticated.

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